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Uniform convexity, uniform non-squareness and von Neumann-Jordan constant for Banach spaces

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Introduction

In connection with the famous work [7] of Jordan and von Neumann concerning inner products Clarkson [2] introduced the von Neumann-Jordan (NJ-) constant for Banach spaces X. Despite its fundamental nature very little is known on the NJ-constant by now. This note is a résumé of some recent results of the authors [12, 15] on the NJ-constant especially concerning some geometrical properties of Banach spaces such as uniform convexity, uniform non-squareness, and also super-reflexivity.

1. Definitions and prelimainary results

The von Neumann-Jordan constant for a Banach space X ([2]), we denote it by $C_{NJ}(X)$, is defined to be the smallest constant C for which

$$\frac{1}{C} \leqq \frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leqq C \tag{1.1}$$

hold for all x, y $\in$ X with (x, y) $\neq$ (0, 0). (Note that the left and right-hand side inequalities in (1.1) are equivalent; indeed, put $x+y=u$, $x-y=v$). The following facts are easily seen:

A. Proposition. (i) $C_{NJ}(X) = 2^{2/t-1}$, $1 \leqq t \leqq 2$, if and only if

$$\| A : l_2^2(X) \to l_2^2(X) \| = 2^{1/t},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $l_2^2(X)$ denotes the X-valued l_2^2-space; and hence

(ii) $C_{NJ}(X') = C_{NJ}(X)$, where X' is the dual space of X. (This was observed for L_p in Clarkson [2].)

Let us recall some classical and recent results in [7], [2], and [10], [9] (see also [11]), where the NJ-constant is calculated for some concrete Banach spaces:

B. Theorem. (i) $1 \leqq C_{NJ}(X) \leqq 2$ for any Banach space X; and $C_{NJ}(X) = 1$ if and only if X is a Hilbert space (Jordan and von Neumann [7]).

(ii) Let $1 \leqq p \leqq \infty$. Then, $C_{NJ}(L_p) = 2^{2/t-1}$, where $t = \min\{p, p'\}$, $1/p + 1/p' = 1$ (Clarkson [2]; see also [10]).

(iii) Let $1 \leqq p, q \leqq \infty$. Then, for $L_p(L_q)$ (L_q-valued L_p-space on arbitrary measure spaces), $C_{NJ}(L_p(L_q)) = 2^{2/t-1}$, where $t = \min\{p, q, p', q'\}$, $1/p + 1/p' = 1/q + 1/q' = 1$; and for the Sobolev space $W_p^k(\Omega)$, $C_{NJ}(W_p^k(\Omega)) = 2^{2/t-1}$, where $t = \min\{p, p'\}$ (Kato and Miyazaki [10]).

(iv) For $E = C_c(K)$ resp. $C_b(K)$ (the spaces of continuous functions on a locally compact Hausdorff space K which have compact support resp. are bounded), $C_{NJ}(E) = 2$ (Kato and Miyazaki [9]).

The above results (iii) (in particular, (ii)) can be obtained in a more simple way than [10] by using arguments in Hashimoto, Kato and Takahashi [8; Corollary 3.6; see also Theorem 3.2].

Let us recall some definitions. A Banach space X is called:

(i) *strictly convex* if $\|(x+y)/2\| < 1$ whenever $\|x\| = \|y\| = 1$, $x \neq y$,

(ii) *uniformly convex* provided for each ε $(0 < \varepsilon < 2)$ there exists a $\delta > 0$ such that $\|(x+y)/2\| < 1 - \delta$ whenever $\|x - y\| \geqq \varepsilon$, $\|x\| = \|y\| = 1$,

(iii) *$(2, \varepsilon)$-convex*, $\varepsilon > 0$, (cf. [13]) provided $\min\{\|x+y\|, \|x-y\|\} \leqq 2(1-\varepsilon)$ whenever $\|x\| = \|y\| = 1$,

(iv) *uniformly non-square* ([5]; cf. [1], [3]) if there is a $\delta > 0$ such that there do not exist x and y in the closed unit ball of X for which $\|(x+y)/2\| > 1 - \delta$ and $\|(x-y)/2\| > 1 - \delta$. (Note that uniform non-squareness is equivalent to $(2, \varepsilon)$-convexity.)

A Banach space Y is said to be *finitely representable* in X if for any $\lambda > 1$ and for each finite-dimensional subspace F of Y, there is an isomorphism T of F into X for which

$$\lambda^{-1}\|x\| \leqq \|Tx\| \leqq \lambda\|x\| \quad \text{for all } x \in F.$$

X is said to be *super-reflexive* ([6]; cf. [1], [3], [13], [14]) if no non-reflexive Banach space is finitely representable in X.

Super-reflexive spaces are characterized as those uniformly convexifiable:

C. Theorem (Enflo [4]; cf. [1], [3], [14]). A Banach space X is super-reflexive if and only if X admits an equivalent uniformly convex norm.

2. Uniform convexity, super-reflexivity and von Neumann-Jordan constant

We bigin with the following proposition which will give effective examples later.

2.1 Proposition. Let $\lambda > 1$. Let $X_{2,\lambda}$ be the space l_2 equipped with the norm $\|x\|_{2,\lambda} := \max\{\|x\|_2, \lambda\|x\|_\infty\}$. Then:

(i) $X_{2,\lambda}$ is isomorphic to a Hilbert space and

$$C_{NJ}(X_{2,\lambda}) = \min\{\lambda^2, 2\}.$$

(ii) $X_{2,\lambda}$ is not strictly convex for any $\lambda > 1$.

Proof (sketch). (i) Since

$$\|x\|_2 \leqq \|x\|_{2,\lambda} \leqq \lambda\|x\|_2 \quad \text{for all } x \in X_{2,\lambda},$$

we have

$$\begin{aligned}\|x+y\|^2_{2,\lambda} + \|x-y\|^2_{2,\lambda} &\leqq \lambda^2(\|x+y\|^2_2 + \|x-y\|^2_2)\\ &= \lambda^2 2(\|x\|^2_2 + \|y\|^2_2)\\ &\leqq \lambda^2 2(\|x\|^2_{2,\lambda} + \|y\|^2_{2,\lambda}),\end{aligned}$$

or $C_{NJ}(X_{2,\lambda}) \leqq \lambda^2$. By considering $x = (1/\lambda, 1/\lambda, 0, \ldots)$ and $y = (1/\lambda, -1/\lambda, 0, \ldots) \in X_{2,\lambda}$, we have $C_{NJ}(X_{2,\lambda}) = \min\{\lambda^2, 2\}$.

(ii) To see that $X_{2,\lambda}$ is not strictly convex, take an α satisfying $(1/\lambda)^2 + \alpha^2 \leqq 1$ and $0 < \alpha \leqq 1/\lambda$; then put $x = (1/\lambda, 0, 0, \dots)$ and $y = (1/\lambda, \alpha, 0, \dots)$.

Now, in the following two theorems we see that uniform convexity is nearly characterized by the condition $C_{NJ}(X) < 2$.

2. 2 Theorem. (i) If X is uniformly convex, then $C_{NJ}(X) < 2$; the converse is not true; indeed,

(ii) For any $\varepsilon > 0$ there exists a Banach space X (isomorphic to a Hilbert space) with $C_{NJ}(X) < 1 + \varepsilon$ which is not even strictly convex.

Proof (sketch). (i) Let X be uniformly convex. Let ε be any positive number with $0 < \varepsilon < 2^{1/2}$. Then, there exists a $\delta > 0$ such that $\|x\| \leqq 1$, $\|y\| \leqq 1$ and $\|x - y\| \geqq \varepsilon$ imply

$$\|(x + y)/2\|^2 \leqq (1 - \delta)\{(\|x\|^2 + \|y\|^2)/2\} \tag{2.1}$$

(cf. [1], p. 190). Now, let x and y be any elements in X with $\|x\|^2 + \|y\|^2 = 1$. We first assume that $\|x - y\| \geqq \varepsilon$. Then, using (2. 1), we have

$$\|x + y\|^2 + \|x - y\|^2 \leqq 2(2 - \delta). \tag{2.2}$$

Next, if $\|x - y\| \leqq \varepsilon$, we have

$$\begin{aligned} \|x + y\|^2 + \|x - y\|^2 &\leqq 2(\|x\|^2 + \|y\|^2) + \varepsilon^2 \\ &\leqq 2(1 + \varepsilon^2/2). \end{aligned} \tag{2.3}$$

Consequently, by (2. 2) and (2. 3) we obtain

$$\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} \leqq 1+\max\{1-\delta,\ \varepsilon^2/2\},$$

or $C_{NJ}(E) < 2$.

(ii) By Proposition 2.1, for the spaces $X_{2,\lambda}$ ($\lambda > 1$) we have $C_{NJ}(X_{2,\lambda}) \to 1$ as $\lambda \to 1$, whereas $X_{2,\lambda}$ is not strictly convex.

Although the condition $C_{NJ}(X) < 2$ does not even imply strict convexity for X, it assures the existence of an equivalent norm on X for which X becomes uniformly convex (cf. Theorem C):

2.3 Theorem. Let $C_{NJ}(X) < 2$. Then, X is super-reflexive; the converse is not true.

Proof. Assume $C := C_{NJ}(X) < 2$. Let x and y be any elements in X with $\|x\| = \|y\| = 1$. Then,

$$\begin{aligned}\min_{\varepsilon_i=\pm1} \|\varepsilon_1 x+\varepsilon_2 y\| &\leqq \left\{\frac{1}{2}(\|x+y\|^2+\|x-y\|^2)\right\}^{1/2}\\ &\leqq C^{1/2}(\|x\|^2+\|y\|^2)^{1/2}\\ &= (2C)^{1/2},\end{aligned}$$

that is, X is $(2, \varepsilon)$-convex with some ε, or equivalently uniformly non-square, which implies that X is super-reflexive (James [6]; see also [1], [13]).

For the latter assertion, consider the space $X_{2,\sqrt{2}}$. Indeed, $X_{2,\sqrt{2}}$ is isomorphic to a Hilbert space and hence super-reflexive, whereas $C_{NJ}(X_{2,\sqrt{2}}) = 2$ by Proposition 2.1. (l_1^n and l_∞^n are also such examples.)

2. 4 Definition. Let $\tilde{C}_{NJ}(X)$ be the infimum of all NJ-constants for equivalent norms of X.

Theorems 2. 2 and 2. 3 assert that super-reflexivity is characterized by the condition $\tilde{C}_{NJ}(X) < 2$.

2. 5 Theorem. The following are equivalent:

(i) $\tilde{C}_{NJ}(X) < 2$.

(ii) X is super-reflexive.

(iii) X admits an equivalent uniformly convex norm.

(iv) X admits an equivalent uniformly non-square norm.

(v) X admits an equivalent uniformly smooth norm (cf. [1]).

(vi) X is J-convex (cf. [1]).

For some further conditions quivalent to super-reflexivity, we refer the reader to [1], [3] and [14].

2. 6 Corollary. $\tilde{C}_{NJ}(X) = 2$ if and only if X is not super-reflexive.

3. Uniform non-squareness and von Neumann-Jordan constant

Very recently the authors [15] proved some homogeneous characterizations of uniformly non-square spaces, one of which is similar to a well-known characterization of uniformly convex spaces (we have used it in the proof of Theorem 2. 2!):

3.1 Theorem Let $1 < p < \infty$. For a Banach space X the following are equivalent:

(i) X is uniformly non-square.

(ii) There exist some ε and δ $(0 < \varepsilon, \delta < 1)$ such that $\|x - y\| \geqq 2(1 - \varepsilon)$, $\|x\| \leqq 1$, $\|y\| \leqq 1$ implies

$$\left\|\frac{x + y}{2}\right\|^{p} \leqq (1 - \delta)\frac{\|x\|^{p} + \|y\|^{p}}{2}.$$

(iii) There exists some δ $(0 < \delta < 2)$ such that for any x, y in X,

$$\left\|\frac{x + y}{2}\right\|^{p} + \left\|\frac{x - y}{2}\right\|^{p} \leqq (2 - \delta)\frac{\|x\|^{p} + \|y\|^{p}}{2}.$$

(iv) $\|A : l_p^2(X) \to l_p^2(X)\| < 2.$

We omit the proof, which will appear elsewhere.

Owing to Theorem 3.1 a precise characterization of Banach spaces with NJ-constant less than two is obtained (cf. Theorems 2.2 and 2.3):

3.2 Theorem The following are equivalent:

(i) $C_{NJ}(X) < 2$.

(ii) X is uniformly non-square.

(iii) X is $(2, \varepsilon)$-convex for some $\varepsilon > 0$.

To see this, merely recall Proposition A.

3.3 Corollary. $C_{NJ}(X) = 2$ if and only if X is uniformly square.

3.4 Note. Further investigation on the NJ-constant is made in [12] especially for the spaces having NJ-constant $2^{2/p-1}$, $1 \leqq p \leqq 2$

(the same value of that of L_p-spaces; see Proposition B).

Our results stated in this note are summarized as follows:

reflexive

$\widetilde{C}_{NJ}(X) < 2$

$\Leftrightarrow$ super-reflexive
$\Leftrightarrow$ uniformly convexifiable
$\Leftrightarrow$ J-convex

$C_{NJ}(X) < 2$

$\Leftrightarrow$ uniformly non-square
$\Leftrightarrow$ $(2, \varepsilon)$-convex

uniformly convex

References

[1] B. Beauzamy, Introduction to Banach spaces and their geometry, 2nd Ed., North Holland, 1985.

[2] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue space, Ann. of Math. 38 (1937), 114-115.

[3] D. van Dulst, Reflexive and super-reflexive Banach spaces, Math. Centre Tracts 102, Math. Centrum, Amsterdam, 1978.

[4] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. 13 (1972), 281-288.

[5] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. 80 (1964), 542-550.

[6] R. C. James, Super-reflexive Banach spaces, Canad. J. Math. 24 (1972), 896-904.

[7] P. Jordan and J. von Neumann, On inner products in linear metric spaces, Ann. of Math. 36 (1935), 719-723.

[8] K. Hashimoto, M. Kato and Y. Takahashi, Generalized Clarkson's inequalities for Lebesgue-Bochner spaces, to appear in Bull. Kyushu Inst. Tech., Math. Natur. Sci. 43 (1996).

[9] M. Kato and K. Miyazaki, Remark on generalized Clarkson's inequalities for extreme cases, Bull. Kyushu Inst. Tech., Math. Natur. Sci. 41 (1994), 27-31.

[10] M. Kato and K. Miyazaki, On generalized Clarkson's inequalities for $L_p(L_q)$ and Sobolev spaces, to appear in Math. Japon.

[11] M. Kato and Y. Takahashi, On Clarkson-Boas-type inequalities, RIMS Kokyuroku (Kyoto Univ.) 897 (1995), 46-58.

[12] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, to appear in Proc. Amer. Math. Soc.

[13] J. Kuelbs, Probability on Banach spaces, Marcel Dekker, New York-Basel, 1978.

[14] L. Schwartz, Geometry and probability in Banach spaces, Lecture Notes in Math. No. 852, Springer, 1981.

[15] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, preprint.